ON QUASI-ARTINIAN RINGS

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P. Vámos introduced in [3] a notion for modules dual to that of "finitely generated". The author studied this condition in his thesis, referring to modules which satisfy it as "quasi-Artinian". Since the majority of results presented here are natural generalizations of results for Artinian rings, it seems appropriate to continue this terminology.

We will assume throughout that R denotes an associative ring with identity and that all R-modules will be understood to be unitary left R-modules. For an R-module M recall that a submodule $N \subseteq M$ is said to be *essential* if $N \cap N' \neq 0$ for all non-zero submodules $N' \subseteq M$, and that M is called *Artinian* if it satisfies the descending chain condition for submodules.

Definition. A module ${}_RM$ is said to be *quasi-Artinian* if it contains an essential Artinian submodule.

If $S(M)$ denotes the socle of M (the sum of all simple submodules of M), it can be shown that M is quasi-Artinian if and only if $S(M)$ is essential and finitely generated, so this notion coincides with that of "finitely embedded" in [3]. It is easy to see that a finite direct sum of quasi-Artinian modules is quasi-Artinian and that any non-zero submodule of a quasi-Artinian module is quasi-Artinian.

An R-module M is called *faithful* if for each $0 \neq r \in R$ there exists $m \in M$ such that $rm \neq 0$, and *completely faithful* if the module ${}_RR$ is isomorphic to a direct summand of a direct sum M^n of finitely many copies of M. M is faithful if and only if ${}_RR$ can be embedded in a direct product of copies of M, so it is immediate that a completely faithful module is faithful. The following definition gives an intermediate notion, which was shown by the author in [2] to be dual to the notion of "faithful".

Definition. A module ${}_RM$ is said to be *co-faithful* if for some integer n, M^n has a submodule isomorphic to ${}_RR$.

If the module ${}_RR$ is quasi-Artinian, we say that R is a left *quasi-Artinian* ring. The notion of "co-faithful" can be used to characterize such rings, as the following proposition shows.

PROPOSITION 1. *A ring R is left quasi-Artinian if and only if every faithful left R-module is co-faithful.*

Proof. Assume R is left quasi-Artinian. If M is faithful, and $m \in M$, let $f_m : R \to M$ be the R-homomorphism defined by setting $f(r) = rm$. Since M is faithful, the intersection of the kernels of the homomorphisms f_m is zero, so by Proposition 1* of [3], there is a finite subcollection of these kernels whose intersection is zero. The corresponding homomorphisms then give an embedding of ${}_RR$ into M^n for the corresponding n and M is co-faithful.

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Conversely, assume that every faithful left R-module is co-faithful, and let $Q(R)$ denote the direct sum of representatives of each isomorphism class of injective envelopes of simple R-modules. $Q(R)$ is known to be faithful, and hence by assumption it is co-faithful. Thus ${}_RR$ can be embedded in $Q(R)^n$ for some n, and since R has an identity, this embedding is actually into a direct sum of finitely many injective envelopes of simple modules. Each of these is quasi-Artinian, so ${}_RR$ is also quasi-Artinian.

This characterization of quasi-Artinian rings can be used to show that the generalized quasi-Frobenius rings studied by G. Azumaya in [1] are precisely the left self-injective and left quasi-Artinian rings, just as quasi-Frobenius rings are the left self-injective and left Artinian rings. These generalized quasi-Frobenius rings are characterized by the fact that any faithful left module is completely faithful.

PROPOSITION 2. *Every faithful left R-module is completely faithful if and only if R is left quasi-Artinian and left self-injective.*

Proof. By virtue of Proposition 1, it suffices to show that every co-faithful left R-module is completely faithful if and only if R is left self-injective. The "if" part is clear from the definitions.

On the other hand, suppose that every co-faithful left R-module is completely faithful. Then the injective envelope $E(R)$ of ${}_RR$ is completely faithful, and ${}_RR$ is isomorphic to a direct summand of $E(R)^n$ for some n and hence is injective.

The characterization of quasi-Artinian rings given by Proposition 1 above is also useful in proving the following two generalizations of results from Artinian rings, that is, rings for which ${}_RR$ is an Artinian module. Let $\text{Rad}(R)$ and $\text{rad}(R)$ denote the *Jacobson radical* and *prime radical* of R, respectively. R is called *semi-simple* if $\text{Rad}(R) = 0$ and *semi-prime* if $\text{rad}(R) = 0$. Recall that R is semi-prime if and only if $A \cdot B = 0$ implies $A \cap B = 0$ for any two sided ideals A and B of R. R is called *prime* if the zero ideal is prime, that is, $A \cdot B = 0$ implies $A = 0$ or $B = 0$ for two-sided ideals A and B of R, and *simple* if there are no proper two-sided ideals of R.

PROPOSITION 3. *If R is left quasi-Artinian, the following are equivalent:*

(i) *R is semi-prime,*

(ii) *R is semi-simple and left Artinian.*

Proof. A semi-simple ring is semi-prime, so it is only necessary to show that (i) implies (ii). If $S(R)$ denotes the socle of ${}_RR$ and $\text{Ann}(S(R))$ the left annihilator of $S(R)$, then both are two-sided ideals and $\text{Ann}(S(R)) \cdot S(R) = 0$. Since R is semi-prime, this implies $\text{Ann}(S(R)) \cap S(R) = 0$, and then $\text{Ann}(S(R)) = 0$ since $S(R)$ is essential. Thus $S(R)$ is faithful, and by assumption co-faithful. This shows that ${}_RR$ can be embedded in $S(R)^n$ for some n, so ${}_RR$ is a direct sum of minimal left ideals, and therefore semi-simple and left Artinian.

PROPOSITION 4. *If R is left quasi-Artinian, the following are equivalent:*

(i) *R is prime,*

(ii) *R is simple and left Artinian.*

Proof. Again, we only need to show that (i) implies (ii). Since R is left quasi-Artinian it contains a minimal non-zero left ideal, and the sum of all isomorphic left ideals is a two-sided ideal which we call S. $\text{Ann}(S) \cdot S = 0$ and, since R is prime, $\text{Ann}(S) = 0$, S is faithful and therefore co-faithful. As before, R is a sum of minimal left ideals, all isomorphic, and therefore R is simple and left Artinian.

Proposition 2* of [3] shows that R is left Artinian if and only if R/A is a left Artinian module for all left ideals A. The following result replaces factor modules of R by factor rings.

PROPOSITION 5. *A ring R is left Artinian if and only if R/A is a left quasi-Artinian ring for every two-sided ideal $A \subseteq R$.*

Proof. If R is Artinian, then for any two-sided ideal $A \subseteq R$, R/A is Artinian, and thus quasi-Artinian.

Conversely, assume that R/A is left quasi-Artinian for every two-sided ideal $A \subseteq R$. Using this and Proposition 1* of [3] it is easy to show that R must satisfy the descending chain condition for two-sided ideals. In particular, the descending chain

$$\text{Rad}(R) \supseteq (\text{Rad}(R))^2 \supseteq (\text{Rad}(R))^3 \supseteq \dots$$

must become stationary after a finite number of steps, say n. Let

$$B = (\text{Rad}(R))^n = (\text{Rad}(R))^{n+1} = \dots.$$

We claim $B = 0$. Suppose not. Then let $A = B \cap \{r : r \in R \text{ and } Br = 0\}$. A is a two-sided ideal since both B and the right annihilator of B are two-sided. Furthermore, A is properly contained in B since $B^2 = B$. By assumption, R/A is left quasi-Artinian and B/A must contain a minimal non-zero left ideal since B/A is non-zero. Let C denote the inverse image of this minimal left ideal in R. Then $A \subsetneqq C \subseteq B$, so there exists an element $0 \neq c \in C$ such that $Bc \neq 0$. But $Bc \subseteq C$ since C is a left ideal of R, and $B(Bc) = B^2 c = Bc \neq 0$, so $Bc \nsubseteq A$. Since C/A is minimal, we must have $Bc \equiv C \pmod{A}$. Therefore there exists an element $b \in B$ such that $c - bc \in A$, and $b \in \text{Rad}(R)$ implies that $1-b$ has a left inverse, say $b'(1-b) = 1$. Then $c = b'(1-b)\,c = b'(c-bc)$. But $c - bc \in A$, so $c \in A$ and this contradicts the fact that $Bc \neq 0$. We conclude that $B = 0$, or equivalently, that $\text{Rad}(R)$ is nilpotent.

$R/\text{Rad}(R)$ is semi-simple, and by assumption left quasi-Artinian. By Proposition 3 it is left Artinian, so every left $R/\text{Rad}(R)$-module is a direct sum of simple modules. In particular, this is true for $(\text{Rad}(R))^i/(\text{Rad}(R))^{i+1}$, for $i = 1, 2, \dots$, so as an R-module each of these is a direct sum of simple R-modules. Now regarding $(\text{Rad}(R))^i/(\text{Rad}(R))^{i+1}$ as a left ideal of the left quasi-Artinian ring $R/(\text{Rad}(R))^{i+1}$, it is a sum of minimal left ideals, and therefore contained in the socle of $R/(\text{Rad}(R))^{i+1}$, which has a composition series.

This shows that for $i = 1, 2, \dots$, $(\text{Rad}(R))^i/(\text{Rad}(R))^{i+1}$ has a composition series as a left $R/(\text{Rad}(R))^{i+1}$-module and therefore as a left R-module. We have already shown that $\text{Rad}(R)$ is nilpotent, so this implies that ${}_R R$ has a composition series and R is left Artinian.

Finally, we give a result useful in constructing examples of commutative quasi-Artinian rings. If R is commutative and M is an R-module, the "*idealization*" of

M is defined as follows: Consider the ring of two-by-two matrices of the form $\begin{pmatrix} r & 0 \\ m & r \end{pmatrix}$ where $r \in R$ and $m \in M$. We shall call this ring $R_M{}^*$. M is R-isomorphic to the obvious ideal in $R_M{}^*$ and with this identification M is an ideal whose subideals are just the R-submodules of M.

PROPOSITION 6. *Let R be a commutative ring and M be a faithful R-module. Then $R_M{}^*$ is quasi-Artinian if and only if M is a quasi-Artinian R-module.*

Proof. If $R_M{}^*$ is quasi-Artinian, then M is quasi-Artinian as an $R_M{}^*$-submodule of $R_M{}^*$. Since the R-submodules of M coincide with the $R_M{}^*$-submodules, ${}_RM$ is quasi-Artinian.

Conversely, suppose that M is a faithful quasi-Artinian R-module. Let S be the socle of M and S^* be the corresponding ideal of $R_M{}^*$ consisting of all elements of the form $\begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$ with $s \in S$. S^* must be finitely generated, so it suffices to show that S^* is an essential ideal.

Let $\begin{pmatrix} r_1 & 0 \\ m_1 & r_1 \end{pmatrix}$ be a non-zero element of $R_M{}^*$. We will show that the ideal generated by this element has non-zero intersection with S^*. If $r_1 = 0$, then $m_1 \neq 0$ and since S is essential in M there exists $r_2 \in R$ with $0 \neq r_2 m_1 \in S$. Then

$$0 \neq \begin{pmatrix} 0 & 0 \\ r_2 m_1 & 0 \end{pmatrix} = \begin{pmatrix} r_2 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ m_1 & 0 \end{pmatrix} \in S^*.$$

If $r_1 \neq 0$ then since M is faithful there exists $m_2 \in M$ with $r_1 m_2 \neq 0$. But then there exists $r_2 \in R$ such that $0 \neq r_2 r_1 m_2 \in S$. Therefore

$$0 \neq \begin{pmatrix} 0 & 0 \\ r_2 m_2 & 0 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ m_1 & r_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r_2 r_1 m_2 & 0 \end{pmatrix} \in S^*$$

This shows that S^* is essential, completing the proof.

References

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